IIT JEE Mathematics

Mathematics is the most scoring subject in IIT JEE. Although Physics, Chemistry and Mathematics have equal weight age in the examination but Mathematics has an edge over other two in number of ways. For example mathematics is considered as the rank deciding subject; whenever there is a tie up of overall score between 2 or more candidates often mathematics score decided the AIR in IIT JEE. Conventionally mathematics as forms the backbone as it is used in the other two subjects as well. Topics like calculus, differential equations, and quadratic equations are used indiscriminately in almost all the questions on physics and chemistry in IIT JEE.

Mathematics is scoring as well – topics like probability, differential calculus, trigonometry, straight lines and circles in coordinate geometry, permutations and combinations in algebra are always easy to crack in IIT JEE. If prepared well these topics can help one get a good AIR without too much of effort. The success mantra for IIT JEE is to “practice, learn and solve”. Gone are the times when students used to spend 2 hours solving a question? Now the time is to get answers quickly and choose the right answer from given choices in objective mathematics questions. So if you have practiced all types of questions thoroughly it would help you quickly reach to the answer rather than figuring out how to crack the question. The more your practice and learn the typical methods of solving different types of questions the faster you can crack IIT JEE mathematics questions.

IIT Math Help Topics:

Algebra | Trigonometry | Analytic Geometry | Differential Calculus |Integral Calculus | Vectors

Algebra

Algebra is one of the building blocks of Mathematics in IIT JEE examination. While preparing for IIT JEE, it is this portion where an aspirant begins. Though Algebra begins with Sets and Relations but we seldom get any direct question from this portion. Functions can be said to be a prerequisite to Calculus and hence it is critical in IIT JEE preparation. Sequence and series is one other section which is mixed with other concepts and then asked in the examination. Quadratic equation fetches direct questions too and is also easy to grasp. Binomial Theorem is also a marks fetching topic as the questions on this topic is quite easy. Permutations and Combinations along with Probability is the most important section in Algebra. IIT JEE exam fetches a lot of questions on them. Those who get good IIT JEE rank always do well in this section. Complex Numbers are also important as this fetches question in the IIT JEE exam almost every year. Matrices and Determinant mostly give direct question and there are no twist and turns in the questions based on them.

Topics covered under IIT JEE Algebra are:-

1. Set theory

SET

A set is a well-defined collection of objects or elements. Each element in a set is unique. Usually but not necessarily a set is denoted by a capital letter e.g., A, B, ..., U, V etc. and the elements are enclosed between brackets { }, denoted by small letters a, b, ..., x, y etc. For example:

A = Set of all small English alphabets
= {a, b, c, ..., x, y, z}

B = Set of all positive integers less than or equal to 10
= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

R = Set of real numbers
= {x : - ∞ < x < ∞}

The elements of a set can be discrete (e.g. set of all English alphabets) or continuous (e.g. set of real numbers). The set may contain finite or infinite number of elements. A set may contain no elements and such a set is called Void set or Null set or empty set and is denoted by Φ(phi). The number of elements of a set A is denoted as n(A) and hence n(Φ) = 0 as it contains no element.

Union of Sets

Union of two or more sets is the best of all elements that belong to any of these sets. The symbol used for union of sets is 'U'

i.e. AUB = Union of set A and set B
= {x : x ε A or x ε B (or both)}

e.g. If A = {1, 2, 3, 4} and B = {2, 4, 5, 6} and C = {1, 2, 6, 8} then AUBUC = {1, 2, 3, 4, 5, 6 8}.

Intersection of Sets

It is the set of all of the elements, which are common to all the sets. The symbol used for intersection of sets is '∩'.

i.e. A∩B = {x : x ε A and x ε B}
e.g. If A = {1, 2, 3, 4} an B = {2, 4, 5, 6} and C = {1, 2, 6, 8}, then A∩B∩C = {2}.
Remember that n(AUB) = n(A) + n(B) -(A∩B).

Difference of Two Sets

The difference of set A to B denoted as A - B is the set of those elements that are in the set A but not in the set B i.e. A - B = {x : x ε A and x ∉ A}.
In general A-B≠B-A
e.g. If A = {a, b, c, d} and B = {b, c, d} then A - B = {a, d} and B - A = {e, f}.

Subset of a Set

A set A is said to be a subset of the set B is each element of the set A is also the element of the set B. The symbol used is '$\subset$' i.e. $A \subseteq B$ <=> (x ε A => x ε B).

Each set is a subset of its own set. Also a void set is a subset of any set. If there is at least one element in B which does not belong to the set A, then A is a proper subset of set B and is denoted as $A \subset B$.

e.g. If A = {a, b, c, d} and B = {b, c, d} then $B \subset A$ or equivalently $A \supset B$ (i.e. A is a super set of B).

Equality of Two Sets

Sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$ and we write A = B.

Universal Set

As the name implies, it is a set with collection of all the elements and is usually denoted by U. e.g. set of real numbers R is a universal set whereas a set A = [x : x < 3} is not a universal set as it does not contain the set of real numbers x > 3. Once the universal set is known, one can define the Complementary set of a set as the set of all the elements of the universal set which do not belong to that set. e.g. If A = {x : x < 3 then $\overline{A}$ (or A^c) = complimentary set of A = {x : x > 3}.

Hence we can say that A U $\overline{A}$ = U i.e. Union of a set and its complimentary is always the Universal set and A ∩ $\overline{A}$ = f i.e. intersection of the set and its complimentary is always a void set. Some of the useful properties of operation on sets are as follows:

- $\overline{\overline{A}}$ (or (Ac)c) = A, $A \cap Ac = \phi$ and $A \cup Ac = U$
- $A \cup \phi = A$ and $A \cap \phi = \phi$
- $A \cup U = U$ and $A \cap U = A$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Illustration:

If A = {a, b, c} and B = {b, c, d} then evaluate A ∪ B, A ∩ B, A - B and B - A.

Solution:

A U B = {x : x ε A or x ε B} = {a, b, c, d}
A ∩ B = {x : x ε A or x ε B} = {b, c}

A - B = {x : x ε A and x ∉ B} = {a}

B - A = {x : x ε B and x ∉ A} = {d}

Natural Numbers

The numbers 1, 2, 3, 4 .... are called natural numbers, their set is denoted by N.
Thus N = {1, 2, 3, 4, 5 ......}

Integers

The numbers .....-3, -2, -1, 0, 1, 2, 3 .... are called integers and the set is denoted by I or Z.
Thus I (or Z) = {.... -3, -2, -1, 0, 1, 2, 3 ....}. Including among set of integers are:

- Set of positive integers denoted by 1+ and consists of {1, 2, 3,....} (≡N)
- Set of negative integers, denoted by 1- and consists of {..., 3, -2, -1}
- Set of non-negative integers {0, 1, 2,...} called as set of whole numbers
- Set of non-positive integers {..., -3, -2, -1, 0}

Rational Numbers

All numbers of the form p/q where p and q are integers and q ≠ 0, are called rational numbers and their set is denoted by Q.

Thus Q = {p/q : p,q ε I and q≠0 and HCF of p,q, is 1}. It may be noted that every integer is a rational number since it can be written as p/1. It may also be noted that all recurring decimals are rational numbers. e.g., p = 0.3 = 0.33333.....
And 10p - p = 3 => 9p = 3 => p = 3/9 => p = 1/3, which is a rational number.

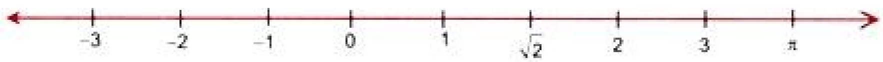
Irrational Numbers

There are numbers which cannot be expressed in p/q form. These numbers are called irrational numbers and their set is denoted by Q^c (i.e. complementary set of Q) e.g. √2, 1 + √3, p etc. Irrational numbers cannot be expressed as recurring decimals.

Real Numbers

The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus R = Q ∪ Q^c.

It may be noted that N $\subset$ I $\subset$ Q $\subset$ R. The real numbers can also be expressed in terms of position of a point on the real line. The real line is the number line where the position of a point relative to the origin (i.e. 0) represents a unique real number and vice versa.



Real Number

All the numbers defined so far follow the order property i.e. if there are two numbers a and b then either a < b or a = b or a > b.

INTERVALS

Intervals are basically subsets of R and are of very much importance in calculus as you will get to know shortly. If there are two numbers a, b ε R such that a < b, we can define four types of intervals as follows:

- Open interval: (a, b) = {x : a < x < b} i.e. end points are not included
- Closed interval: [a, b] = {x : a $\leq$ x $\leq$ b} i.e. end points are also included.
- This is possible only when both a and b are are finite.
- Open-closed interval : (a, b] = {x : a < x $\leq$ b}
- Closed-open interval : [a, b) = {x : a $\leq$ x < b}

The infinite intervals are defined as follows:

- (a, ∞) = {x : x > a}
- [a, ∞) = {x : x > a}
- (-∞, b) = {x : x < b}
- (-∞, b] = {x : x < b}

Intervals are particularly important in solving inequalities or in finding domains etc.

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2. **Inequalities**

The following are some very useful points to remember:

- $a \leq b \Rightarrow$ either $a < b$ or $a = b$
- $a < b$ and $b < c \Rightarrow a < c$
- $a < b \Rightarrow a + c < b + c \ \forall \ c \in R$
- $a < b \Rightarrow -a > -b$ i.e. inequality sign reverses if both sides are multiplied by a negative number
- $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$
- $a < b \Rightarrow ma < mb$ if $m > 0$ and $ma > mb$ if $m < 0$
- $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$
- $(a+(1/2)) \geq 2 \ \forall \ a > 0$ and equality holds for $a = 1$
- $(a+(1/2)) \leq -2 \ \forall \ a > 0$ and equality holds for $a = -1$

SINOSODIAL CURVE METHOD

In order to solve inequalities of the form $(P(x)/Q(x)) \geq 0$, $(P(x)/Q(x)) \leq 0$, where $P(x)$ and $Q(x)$ are polynomials, we use the following method:

If x_1 and x_2 ($x_1 < x_2$) are two consecutive distinct roots of a polynomial equation, then within this interval the polynomial itself takes on values having the same sign. Now find all the roots of the polynomial equations $P(x) = 0$ and $Q(x) = 0$. Ignore the common roots and write

$$(P(x)/Q(x)) = f(x) \ (((x-\alpha_1)(x-\alpha_2)(x-\alpha_3)....(x-\alpha_n))/((x-\beta_1)(x-\beta_2)(x-\beta_3)....(x-\beta_m)))$$

where $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m$ are distinct real numbers. Then $f(x) = 0$ for $x = \alpha_1, \alpha_2, ..., \alpha_n$ and $f(x)$ is not defined for $x = \beta_1, \beta_2, ..., \beta_m$. Apart from these $(m + n)$ real numbers $f(x)$ is either positive or negative. Now arrange $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m$ in an increasing order say $c_1, c_2, c_3, c_4, c_5, ..., c_{m+n}$. Plot them on the real line. An draw a curve starting from right of c_{m+n} along the real line which alternately changes its position at these points. This curve is known as the sinosodial curve.

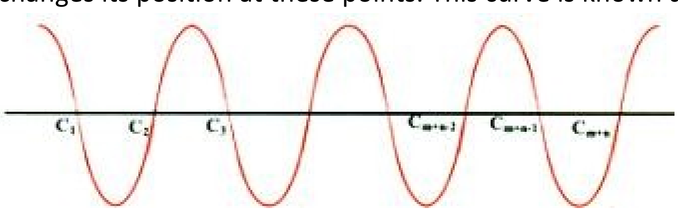
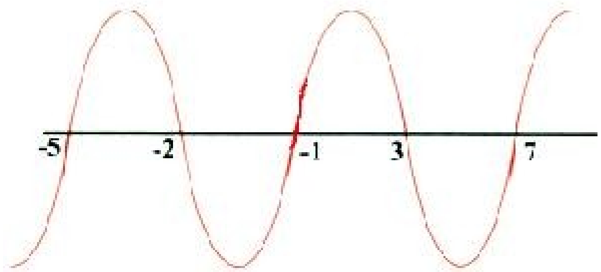


Illustration: Let $f(x) = ((x-3)(x+2)(x+5))/((x+1)(x-7))$. Find the intervals where $f(x)$ is positive or negative.

Solution: The relevant sinosodial curve of the given function is



$f(x) > 0 \ \forall \ x \ \varepsilon \ (-5, -2) \cup (-1, 3) \cup (7, \infty)$ and

$f(x) < 0 \ \forall \ x \ \varepsilon \ (-\infty, -5) \cup (-2, -1) \cup (3,)$.

Exercise

(i) Let $f(x) = (x^2-3x+2)/(x^2-1)$. Find the intervals where $f(x)$ is negative.

(ii) If $((x-1)(x-2)^2(x-3)^3)/((x-4)^2(x-5)^6) > 0$, then find the values of x.

LOGARITHM

* The expression $\log_b a$ is meaningful for $a > 0$ and for either $0 < b < 1$ or $b > 1$.
* $a = b^{\log_b a}$
* $\log_a b = \log_c b / \log_c a$
* $\log_b a = 1/\log_a b$ provided both a and b are non-unity

$$\Rightarrow \begin{matrix} a_1 \geq a_2 > 0 \\ 0 > a_1 \leq a_2 \end{matrix} \text{ if } \begin{matrix} if & b > 1 \\ & 0 < b < 1 \end{matrix}$$

* $\log_b a_1 > \log_b a_2$

ABSOLUTE VALUE

Let $x \ \varepsilon \ R$. Then the magnitude of x is called it's absolute value and is, in general, denoted by $|x|$. Thus $|x|$ can be defined as,

$$|x| = \begin{cases} -x & x \leq 0, \\ x, & x > 0. \end{cases}$$

Note that x = 0 can be included either with positive values of x or with negative values of x. As we know all real numbers can be plotted on the real number line, $|x|$ in fact represents the distance of number 'x' from the origin, measured along the number-line. Thus, $|x| \geq 0$. Secondary, any point 'x' lying on the real number line will have its coordinate as (x, 0). Thus its distance from the origin is $\sqrt{x^2}$.

Hence $|x| = \sqrt{x^2}$. Thus we can defined $|x|$ as $|x| = \sqrt{x^2}$ or $|x| = \max(x, -x)$

e.g. if x = 2.5, then |x| = 2.5, if x = 3.8 then |x| = 3.8.

Basic Properties of |x|

* $||x|| = |x|$
* $|x| > a \Rightarrow x < a$ or $x < -a$ if $a \in R+$ and $x \in R$ if $a \in R-$
* $|x| < a \Rightarrow -a < x < a$ if $a \in R+$ and no solution if $a \in R- \cup \{0\}$
* $|x + y| \leq |x| + |y|$
* $|x - y| \geq |x| \sim |y|$
* The last two properties can be put in one compact form namely,

  $|x| \sim |y| \leq |x \pm y| \leq |x| + |y|$
* $|xy| = |x| \; |y|$
* $|x/y| = |x/y|$ $y \neq 0$

Illustration: Solve the inequality for real values of x: $|x - 3| > 5$.

Solution: $|x - 3| > 5 \Rightarrow x - 3 < -5$ or $x - 3 > 5$

$\Rightarrow x < -2$ or $x > 8 \Rightarrow x \in (-\infty, -2) \cup (8, \infty)$.

GREATEST INTEGER AND FRACTIONAL PART

Let $x \in R$ then [x] denotes the greatest integer less than or equal to x and {x} denotes the fractional part of x and is given by {x} = x - [x]. Note that $0 < \{x\} < 1$.

e.g. $x = 2.69 \Rightarrow 2 < x < 3 \Rightarrow [x] = 2$, $x = -3.63 \Rightarrow -4 < x < -3 \Rightarrow [x] = -4$

It is obvious that if x is integer, then [x] = x.

Basic Properties of greatest Integer and Fractional Part

* [[x]] = [x]x, [{x}] = 0, {[x]} = 0
* $x - 1 < [x] \leq x$, $0 \leq \{x\} < 1$
* [n + x] = n + [x] where $n \in I$
* [x] + [-x] $= \begin{cases} 0, & if\ x \in integer \\ -1, & if\ x \notin integer \end{cases}$
* {x} + {-x} $= \begin{cases} 0, & x \in integer \\ 1, & x \notin integer \end{cases}$

$$[x+y] = \begin{cases} [x] + [y], & if\{x\} + \{y\} < 1 \\ [x] + [y] + 1, & if\ \{x\} + \{y\} \geq 1 \end{cases}$$

* [x+y] Hence $[x] + [y] < [x + y] \leq [x] + [y] + 1$

* $[[x]/n] = [x/n]$, $n \in N$, $x \in R$

Illustration:

If $y = 3[x] + 1 = 2[x - 3] + 5$, then find the value of $[x + y]$.

Solution:

We are given that $3[x] + 1 = 2([x] -3) + 5$

=> $[x] = -2$ => $y = 3(-2) + 1 = -5$.

Hence $[x + y] = [x] + y = -2 - 5 = -7$.

Illustration:

Solve the equation $|2x-1| = 3[x] + 2\{x\}$ for x.

Solution:

Case I: For $x < 1/2$, $|2x -1| = 1 - 2x$ => $1 - 2x = 3[x] + 2\{x\}$

=> $1 -2x = 3(x - \{x\}) + 2\{x\}$ => $\{x\} = 5x - 1$.

Now $0 \leq \{x\} < 1$ => = $0 \leq 5x - 1 < 1$

=> $1/5 \leq x < 2/5$ => $[x] = 0$ => $x = \{x\}$

=> $x = 5x - 1$ => $x = 1/4$, which is a solution.

Case II: For > ½, $|2x - 1| = 2x - 1$

=> $2x - 1 = 3[x] + 2\{x\}$ => $2x - 1 = 3(x - \{x\}) + 2\{x\}$.

$\{x\} = x + 1$

Now $0 < \{x\} < 1$ => $0 < x + 1 < 1$ => $-1 < x < 0$

Which is not possible since $x > 1/2$.

Hence $x = 1/4$ is the only solution

3. **Cartesian Product**

Let A and B are two non-empty sets. The Cartesian product A × B of these sets is defined as the set f all ordered pairs (a, b) such that a ε A and b ε B. For example If A = {2, 3. 4} and B = {4, 9, 16} then A × B = {(2, 9), (2, 16), (3, 9), (3, 4), (3, 16), (4, 9), (4, 4), (4, 16)}. Out of these ordered paired elements of some are related with each other (In above example the second elements of a bold marked ordered pair is the square of the first. Such ordered pairs are called related ordered pairs.

A subset 'f' of the Cartesian product A X B is called a function from A to B if and only if to each 'a' ε A, there exists a unique 'b' in B such that (a, b) ε f. Thus the function from A to B can be described as the set of ordered pairs (a, b) such that a ε A and b ε B and for each 'a' there is a unique 'b'. This function may be written as:

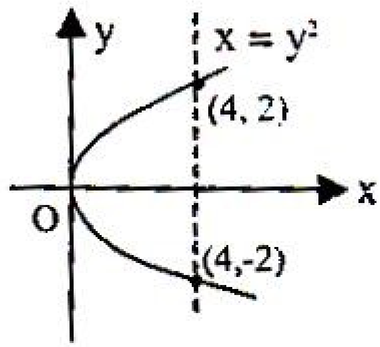
f : A → B or A → B

Thus, a relation from A to B is a function if and only if

(i) to each a ε A, there exists a unique 'b' n B such that (a, b) ε f

(ii) (a_1, b_1) ε f and (a_1b_2), ε f => $b_1 = b_2$.

Graphically, if f(x) is plotted in the y axis against x and if a line parallel to y axis cuts f(x) at more than one point then f(x) does not fulfill the requirement of a function, because for same value of x, you will have two values of y. (see fig. 1)



$x = y^2$ is not a function, because of x = 4, y = $\underline{+}2$.

e.g. If A = [a, b, c} and B = {d, e, f}

Then f = {(a, d), (b, d), (c, e)} is a function from A to B but g = {(a, d), (a, e)} is not a function from A to B.

The unique element b ε B assigned to a ε A is called the image of 'a' under f for value of f at 'a'. 'a' is called the pre-image of 'b'. Also 'a' is called theindependent variable, and 'b' is called the dependent variable.

4. **<u>Introduction to Functions</u>**

Definition of Function: Domain, Co-domain and Range

Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write f:X → Y where
y = f(x), x ε X and y ε Y. We say that 'y' is the image of 'x' under 'f' (or x is the pre image of y).
Two things should always be kept in mind:

(i) A mapping f: $X \rightarrow Y$ is said to be a function if each element in the set X has its image in set Y. It is possible that a few elements in the set Y are present which are not the images of any element in set X.
(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can't be multi-valued (A mapping that is multi-valued is called a relation from X to Y)

- Set 'X' is called the **domain of the function** 'f'.
- Set 'Y' is called the **co-domain of the function** 'f'.
- Set of images of different elements of set X is called the range of the function 'f'. It is obvious that range could be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of the set X (of course these elements of co-domain will not be included in the range). Range is also called domain of variation.

The set of values for which a function is defined is called the domain of the function. The range of the function is the set of all images of domain of f. In above example, the set A is the domain of the function f. B is not range but the co-domain of the function. The **range is the subset of the co-domain**. The domain and the range of a function may be an interval, open, closed, semi-closed or semi-open i.e. the domain may be an interval of any of the following types.

If $x \in [q, p]$, then $\{x : q \leq x \leq p\}$
If $x \in (-\infty, p]$, then $\{x : x \leq p\}$
If $x \in (-\infty, p)$, then $\{x : x < p\}$
If $x \in]-\infty, p[$, then $\{x : x < p\}$
If $x \in (p, \infty]$, then $\{x : x > p\}$
If $x \in (-\infty, \infty)$, then $\{x : x \in R\}$
If $x \in \{p, q\}$, then $\{x : x = p \text{ or } x\ q\}$

Domain of function 'f' is normally represented as Domain (f). Range is represented as Range (f). Note that sometimes domain of the function is not explicitly defined. In these cases domain would mean the set of values of 'x' for which f(x) assumes real values. e.g. if y = f(x) then Domain (f) = {x : f(x) is a real number}.
e.g. Let X = {a, b, c}, Y = {x, y, z}. Suppose f(a) = y, f(a) = x, f(b) = y, f(c) = z. Then f is not a function of X into Y since $a \in X$ has more than one f-images in Y.
On the other hand, if we set f(a) = x, f(b) = x and f(c) = x, then f:X $\rightarrow$ is a function since each element in X has exactly one f-image in Y.
Consider the following examples:

(i) Let X = R, Y = R and $y = f(x) = x^2$.

Then $f : X \rightarrow Y$ is a function since each element in X has exactly one f-image in Y. The range of $f = \{f(x) : x \in X\} = \{x^2 : x \in R\} = [0, \infty)$.

(ii) Let $X = R^+$, $Y = R^+$ and $y = \sqrt{x}$. Then $f : X \rightarrow Y$ is a function. The range of f is R^+

(iii) Let X = R, Y = R and $y^2 = x$. Her $f(x) = +\sqrt{x}$ i.e. f is not a function of X into Y since each $x > 0$ has two f-images in Y, and further, each $x < 0$ has no f-image in Y.

We are primarily interested in functions whose domain and ranges are subsets of real numbers. Such functions are often called Real Valued functions.

e.g. Let the function f be defined by $f(x) = 1/\sqrt{(2x+6)}$.

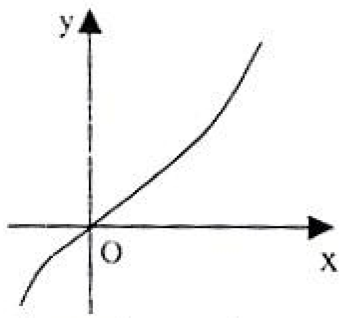
In this formula we must have $2x + 6 > 0$ and therefore $x > -3$. Therefore, the domain of f is $(-3, \infty)$, the range of f = $(0, \infty)$. Thus we have the function $f : (-3, \infty) \rightarrow (0, \infty)$ defined by $f(x) = 1/\sqrt{(2x+6)}$.

Let the function f be defined by $f(x) = x/((x-1)(x-2))$. The formula makes sense for all values of x except x = 1 and x = 2. Therefore, the domain of f is R - {1, 2}

5. **Functions- one-one, many-one, into, onto**

Functions can be classified according to their images and pre-images relationships. Consider the function $x \rightarrow f(x) = y$ with the domain A and co-domain B.

If for each x ε A there exist only one image y ε B and each y ε B has a unique pre-image x ε A (i.e. no two elements of A have the same image in B), then f is said to be one-one function. Otherwise f is many-to-one function.

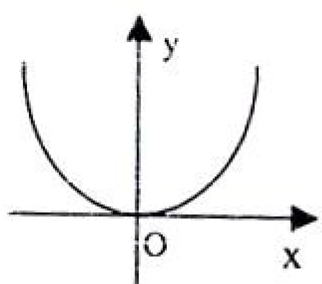


Graph of $x \rightarrow x^3$

e.g. $x \rightarrow x_3$, x ε R is one-one function

while $x \rightarrow x_2$, x ε R is many-to-one function. (see figure above)

e.g. $x = + 2$, $y = x_2 = 4$



Graph of $x \rightarrow x^2$

Graphically, if a line parallel to x axis cuts the graph of f(x) at more than one point then f(x) is many-to-one function and if a line parallel to y-axis cuts the graph at more than one place, then it is not a function.

For a one-to-one function

If $x1 \neq x_2$ then $f(x_1) \neq f(x_2)$

or if $(x_1) = f(x_2) \Rightarrow x_1 = x_2$

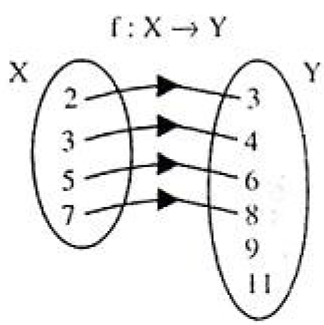
One-to-one mapping is called injection (or injective).

Mapping (when a function is represented using Venn-diagrams then it is called mapping), defined between sets X and Y such that Y has at least one element 'y' which is not the f-image of X are called into mappings.

Let a function be defined as: $f : X \rightarrow Y$

Where X = {2, 3, 5, 7} and Y = {3, 4, 6, 8, 9, 11}

The mapping is shown in the figure below.



Clearly, element 9 and 11 of Y are not the f-image of any of $x \varepsilon X$

So the mapping is into-mapping

Hence for into mappings:

$f[X\} \subset Y$ and $f[X] \neq Y$. => $f[X] \not\subseteq Y$ that is range is not a proper subset of co-domain.

The mapping of 'f' is said to be onto if every element of Y is the f-image of at least one element of X. Onto mapping are also called surjection.

One-one and onto mapping are called bijection.

Illustration

Check whether $y = f(x) = x^3$; $f : R \rightarrow R$ is one-one/many-one/into/onto function.

We are given domain and co-domain of 'f' as a set of real numbers.

For one-one function:

Let $x_1, x_2 \varepsilon D_f$ and $f(x_1) = f(x_2)$

$\Rightarrow X_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

i.e. f is one-one (injective) function.

For onto-into:

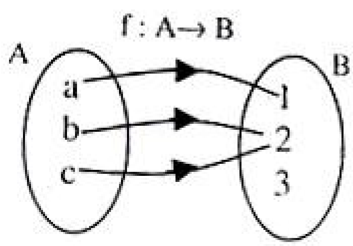
$Lt_{x \to a} \ y = Lt_{x \to a} (x)^3 = \alpha$

$Lt_{x \to a} \ y = Lt_{x \to a} (X)3 = -\alpha$

Therefore $y = x^3$ is bijective function.

Illustration:

What kind of function does the Venn diagram in figure given below represent?



Solution: This many-one into function

Domain = D_f = {a, b, c}

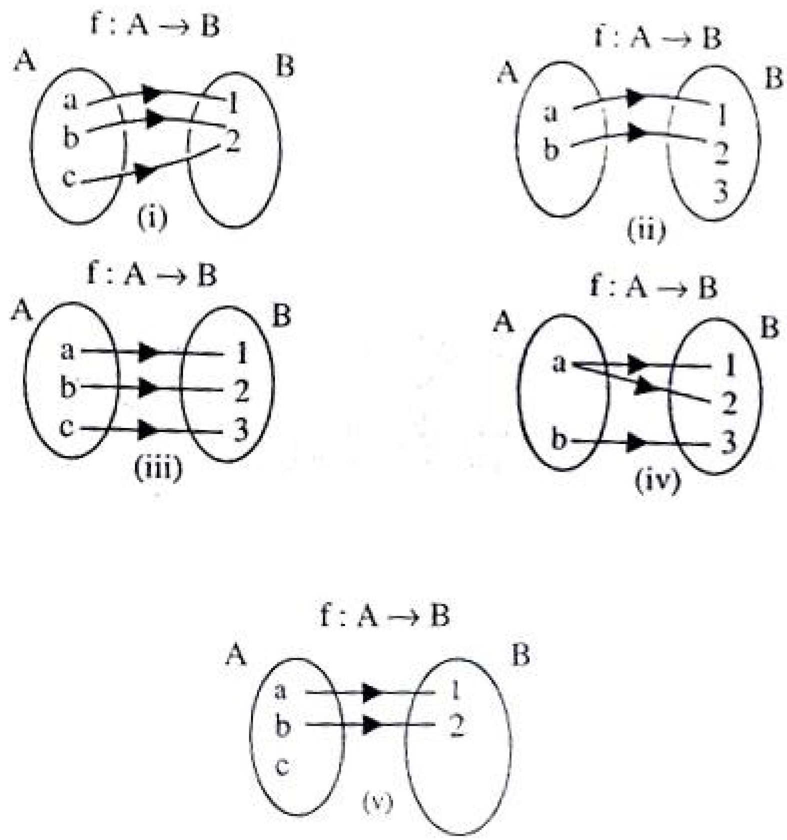
Co-domain = {1, 2, 3}

Range = R_f = {1, 2}

f(a) = 1 ; f(b) = 2; f(c) = 2

Examples

Classify the following functions.



Ans.

(i) Many-one and onto (surjective).

(ii) One-one (injective) and into.

(iii) One-one (injective) and onto (surjective) i.e. Bijective.

(iv) and (v) are not functions.

Examples

1. Given the sets A = {1, 2, 3, 4} and B = {a, b, c} construct a

   (i) Many-one into

   (ii) many-one onto function

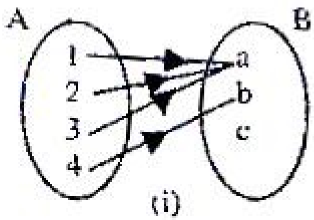
2. Given the sets c = {1, 2, 3} and D = {a, b, c}

   (i) How many one-one onto functions can be constructed.

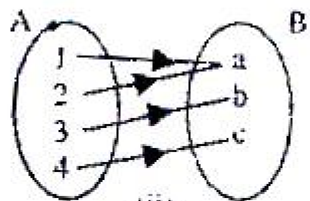
   (ii) How many-one into functions can be constructed.

Ans.1

f : A → B f : A → B



(i)



(ii)

2. (i) 6

(ii) $3^3 - 6 = 21$

Illustration:

What is the domain and range of the following functions?

(a) $y = 3x + 5$ (b) $y = (x^2 + x)/(x^2 - x)$

Domain of $y = f(x)$ is the set of values of x for which y is real and finite.

Range is the set of values of y for which x is real and finite.

Solution:

(a) For all real and finite x, y is also real and finite

Therefore $D_f = R = (-\infty, \infty)$ and $R_f = R = (-\infty, \infty)$

(b) $y = (x(x+1))/(x(x-1)) = (x+1)/(x-1)$, $x \neq 0$

when $x = 0$, y is 0/0 from (i.e. indetermined form)

when $x = 1$, $y = \infty$ (infinite)

Therefore $D_f = R - \{0, 1\}$

also $xy - y = x + 1$

$\Rightarrow x(y - 1) = y + 1$

$x = (y+1)/(y-1)$

when $y = 1$, $x = \infty$ (infinite) $\Rightarrow y \neq 1$

also, for $\neq 0$ $\Rightarrow y \neq -1$

Therefore $R_f = R - \{-1, 1\}$

Illustration:

What is the domain of the following functions?

(a) $y = \sqrt{(x-1)(3-x)}$ (b) $\sqrt{((x-1)(x-5))/(x-3)}$ (c) $y = \sqrt{\sin x}$

Solution:

(a) y is real and finite if $(x - 1)(3 - x) \geq 0$

or $(x - 1)(x - 3) \leq 0$

i.e. $x - 1 \leq 0$ and $x - 3 \geq 0$ or $x - 1 \geq 0$ and $x - 3 \leq 0$

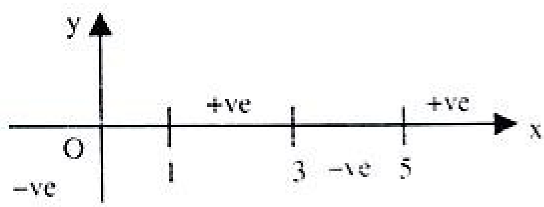
$\Rightarrow x \leq 1$ and $x \geq 3$ $\Rightarrow 1 \leq x \leq 3$

which is not possible $\Rightarrow 1 \leq x \leq 3$

$\Rightarrow D_f = [1, 3]$

(b) Numerator becomes zero for x = 1, x = 5

Denominator becomes zero for x = 3



These three points divide x-axes into four intervals

$(-\infty, 1), (1, 3), (3, 5), (5, \infty)$

Therefore $D_f = [1, 3) \cup [5, \infty)$; at x = 3, we here open interval,

Because at x = 3, y is infinite.

(c) $y = \sqrt{\sin x}$

$\sin x \geq 0 \ \forall \ x \in [2n\pi, (2n + 1)\pi], n \in I$

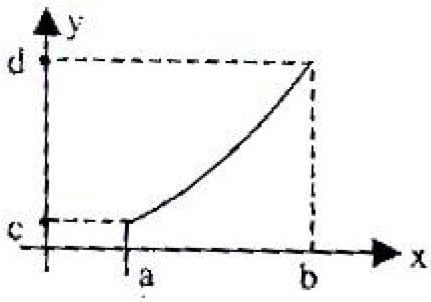
Examples

1. What is domain of the following?

(a) $y = \sqrt{(x-1)(3-x)}$ (b) $y = \sqrt{x \sin x}$ (c) $y = \sin^{-1}((1+x^2)/(2x))$

2. What is domain and range of the following?

(a)



(b) $y = \underline{|x}$

Ans.

1. (a) $D_f = [1, 3)$

(b) $D_f = [-(2n-1)\Pi, -2(n-1)\Pi] \cup [2n\Pi, (2n+1)\Pi]$, $n \in N$

(c) $D_f = \{-1, 1\}$

2. (a) $D_f = [a, b[$ and $R_f = [c, d]$

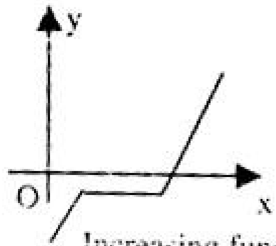
(b) $D_f = \{0, 1, 2, 3, 4, ...\}$

$R_f = \{1, 2, 6, 24, ...\}$

6. **Increasing or decreasing Functions**

The function f is said to be an increasing function in its domain D if

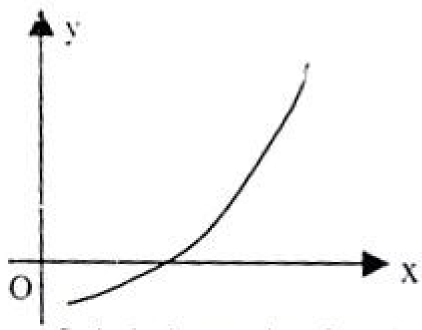
$\forall\ x_2 > x_1 \Rightarrow f(x_2) > f(x_1);\ x_1, x_2 \in D$



Increasing function

However if

$\forall\ x_2 > x_1 \Rightarrow f(x_2) > f(x_1),\ x_1, x_2\ \varepsilon\ D$

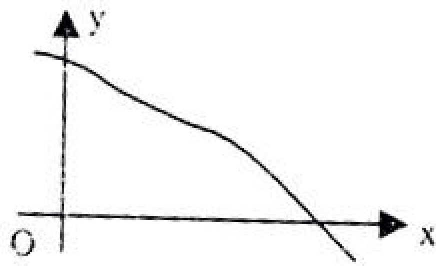


Strictly Increasing function

The function 'f' is said to be strictly increasing

The function 'f' is said to be decreasing function in its domain D if

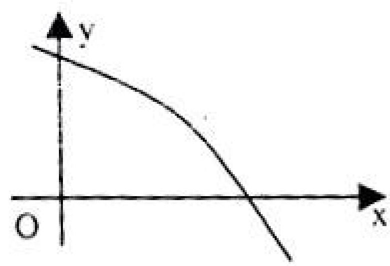
$\forall\ x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1);\ x_1, x_2\ \varepsilon\ D$



Decreasing function

However if

$\forall\ x_2 > x_1 \Rightarrow f(x_2) < f(x_1);\ x_1, x_2\ \varepsilon\ D$



Strictly decreasing function

Then it said to be strictly decreasing.

Strictly increasing and decreasing functions are also called Monotonic Function.

Illustration:

Is $y = 2x + 3$ increasing/decreasing function.

Solution:

Since, $\forall\ x\ \varepsilon\ R,\ y\ \varepsilon\ R$

Therefore $D_f = R$

Let $x_2 > x_1;\ x_1, x_2\ \varepsilon\ R$

$\Rightarrow \quad 2x_2 > 2x_1$

=> $2x_2 + 3 > 2x_1 + 3$

=> $f(x_2) > f(x_1)$

=> $y = f(x) = 2x + 3$ is strictly increasing function.

Examples

1. Are the following function increasing/decreasing?

(a) $y = x3 + 8$ (b) $y = -2x + 4$

Ans. (a) Decreasing

(b) Strictly decreasing

7. **Inverse Functions**

Let $f : X \rightarrow Y$ be a function defined by $y = f(x)$ such that f is both one - one and onto. Then there exists a unique function $g : Y \rightarrow X$ such that for each $y \varepsilon Y$,

$g(y) = x \Leftrightarrow y = f(x)$. The function g so defined is called the inverse of f.

Further, if g is the inverse of f, then f is the inverse of g and the two functions f and g are said to be the inverses of each other. For the inverse of a function to exists, the function must be on-one and onto.

Method to Find Inverse of a Function

If f^{-1} be the inverse of f, then $fof^{-1} = f^{-1}of = I$, where I is an identity function.

$fof^{-1} = I \Rightarrow (fof^{-1}(x)) = I(x) = x$.

Apply the formula of f on $f^{-1}(x)$, we will get an equation in $f^{-1}(x)$ and x.

Solve it to get $f^{-1}(x)$.

Note : A function and its inverse are always symmetric with respect to the line $y = x$.

Illustration: Let $f : R \rightarrow R$ defined by $f(x) = (e^x - e^{-x})/2$. Find $f^{-1}(x)$.

Solution: We have $f(f^{-1}(x)) = x$

=> $(e^{f-1(x)} - e^{-f-1(x)})/2 = x$

=> $e^{2f-1(x)} - 2xe^{f-1(x)} - 1 = 0$

=> $e^{f-1(x)} = x \pm \sqrt{(x^2 + 1)}$.

But negative sign is not possible because L.H.S. is always positive.

Thus $e^{f-1(x)} = x + \sqrt{(x^2 +1)}$. Hence, $f^{-1}(x) = \log(x + \sqrt{(x^2 +1)})$.

We give below some standard functions along with their inverse functions:

FUNCTIONS		INVERSE FUNCTION
1.	f:[0,∞)→[0,∞) defined by $f(x)=x^2$	f^{-1}:[0,∞)→[0,∞) defined by $f^{-1}(x) = \sqrt{x}$
2.	f:[-∏/2,∏/2] →[-1,1] defined by f(x)=sin x	f^1 [-1,1]→[-(∏/2),∏/2] defined by $f^{-1}(x)=\sin^{-1}x$
3.	f:[0,∏]→[-1,1] defined by f(x)=sinx	f^1:[-1,1]→[0,∏] defined by $f^1(x)=\cos^{-1}x$
4.	f:[-∏/2,∏/2] →(-∞,∞) defined by f(x)=tan x	f^1:(-∞,∞)→[-(∏/2),∏/2] defined by $f^1(x)=\tan^{-1}x$
5.	f:(0,∏)→(-∞,∞) defined by f(x) = cot x	f^{-1}:(-∞, ∞)→(0,∏) defined by $f^{-1}(x)=\cot^{-1}x$
6.	f:[0,∏/2)U(n/2,n]→(-∞, -1]U[1,∞) defined by f(x) = sec x	f^{-1}:(-∞,-1]U[1,∞) →[0,∏/2)U(∏/2,∏] defined by $f^{-1}(x) = \sec^{-1}x$
7.	f:[-(∏/2),0)(0,n/2]→(-∞,-1]U[1,∞) defined by f(x) = cosec x	f^{-1}:(-∞,-1]U[1,∞) →[0,-(∏/2))U(0,∏/2] defined by $f^{-1}(x) = \text{cosec}^{-1}x$
8.	f:R → R^+ defined by $f(x) = e^x$	$f^{-1}(x)$:R^+ → R defined by $f^{-1}(x)$ = In x.

8. Invertible Functions

Invertible function

Let us define a function y = f(x): X → Y. If we define a function g(y) such that x = g(y) then g is said to be the inverse function of 'f'.

Think: If f is many-to-one, g : Y → X will not satisfy the definition of a function.

So to define the inverse of a function, it must be one-one.

Further if f : X → Y is into then there must be a point in Y for which there is no x. This again violates the definition of function for 'g' (In fact when f is one tone and onto then 'g' can be defined from range of f to domain of i.e. g : f(X) → X.

Hence, the inverse of a function can be defined within the same sets for x and Y only when it is one-one and onto or Bijective.

Note: A monotonic function i.e. bijection function is always invertible.

Illustration: Let f : R → R be defined as

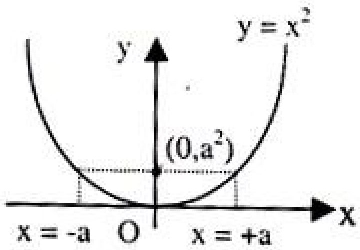
$y = f(x) = x^2$. Is it invertible?

Solution:

No it is not invertible because this is a many one into function

This is many-one because for $x = \pm a$, $y = a^2$, this is into because y does not take the negative real values.

Illustration: Let f : R → [0, α) be defined as $y = f(x) = x^2$. Is it invertible?
(see figure below)



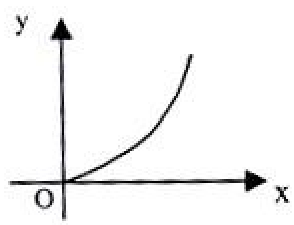
Solution:

No it is not invertible, it because it is many one onto function.

Illustration: Let f : [0, α) → [0, α) be defined as $y = f(x) = x^2$. Is it invertible? If so find its inverse.

Solution:

Yes, it is invertible because this is bijection function. Its graph is shown in figure given below.



Let $y = x^2$ (say f(x))

=> $x = \pm\sqrt{y}$

But x is positive, as domain of f is [0, α)

=> $x = +\sqrt{y}$

Therefore Inverse is $y = \sqrt{x} = g(x)$

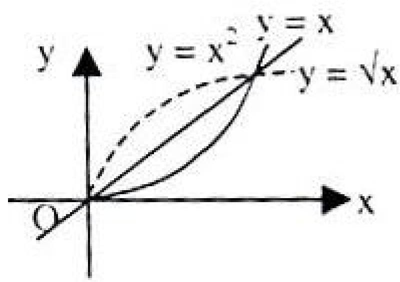


Figure (A)

$f(g(x)) = f(\sqrt{x}) = x, x > 0$

$g(f(x)) = g(x^2) = \sqrt{x^2} = x, x > 0$

i.e. if f and g are inverse of each other then $f(g(x)) = g(f(x)) = x$

Illustration: How are the graphs of function and the inverse function related? These graphs are mirror images of each other about the line $y = x$.

Solution:

Also, if the graph of $y = f(x)$ and $y = f^{-1}(x)$, they intersect at the point where y meet the line $y = x$.

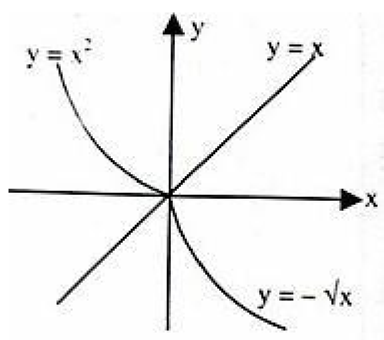


Figure (B)

Graphs of the function and its inverse are shown in figures given above as Figure (A) and (B)

For Figure (A)

$y = f(x) = x^2; f : [0, \infty) \rightarrow [0, \infty)$

Examples

1. Define $y = f(x) = x^2$ in some other ay so that its inverse is possible.

2. What is the inverse of $y = \log_e (x + \sqrt{(x^2 + 1)})$

Ans.1 $f : (-\alpha, 0] \rightarrow [0, \alpha)$

$y = f(x) = x^2$ and its inverse is

$y = -\sqrt{x}$ (Figure B)

Ans.2 $y = (e^x - e^x)/2$

9. **Even and Odd Functions**

A function $f(x) : X \rightarrow Y$ defined such that

$f(-x) = f(x)\ \forall\ x\ \varepsilon\ X$

is called an even function and

if $f(-x) = -d(x)\ \forall\ x\ \varepsilon\ x$, then the function $f(x)$ is called an odd function.

Graphically, an even function is symmetrical w.r.t. y-axis and odd function is symmetrical w.r.t. origin.

Note : In general all functions can be represented as sum of an even function and an odd function.

Let, a function be defined as $y = f(x)$. It can be written as:

$\Rightarrow y = (f(x) + f(-x))/2 + (f(x) - f(-x))/2$

$y = F_1(x) + F_2(x)$

Whereas,

$F_1(-x) = (f(x) + f(-x))/2 = F_1(x)$

And $F_2(-x) = (f(-x) - f(x))/2$

$= -((f(x) - f(x))/2)$

$= -F_2(x)$.

Here $F_1(x)$ is an even function and $F_2(x)$ is an odd function.

Exercise

State whether the following functions are odd or even or neither.

(1) $y = x^3$

(2) $y = x^4$

(3) $y + x + \cos x$

(4) $y = \log e(x + \sqrt{(x^2 + 1)})$

10. **Explicit and Implicit Functions**

If, in a function the dependent variable y can be explicitly written in terms of independent variable x i.e. terms of 'x' must not involve y in any manner then the function is called an explicit function e.g.

$y = x^2 + 1$

$y = \sin x + \cos x$

If the dependent variable y and independent variable x are so convoluted in an equation that y cannot be written explicitly as function of x then f(x) is said to be an implicit function.

e.g. $x^2 + y^2 = \tan^{-1} xy$.